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STRESSES AND DISPLACEMENTS IN AN ELASTIC-PLASTIC WEDGE

by

P. M. Naghdi

March, 1956

Office of Naval Research Project NR-064-408
Contract Nonr-1224 (01)

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SUMMARY

An elastic-perfectly plastic wedge of an incompressible isotropic material in the state of plane strain is considered, where the stress-strain relations of Prandtl-Reuss are employed in the plastic domain. For a wedge (with an included angle β) subjected to a uniform normal pressure on one boundary, the complete solution is obtained which is valid in the range $0<\beta<\frac{\pi}{2}$; this latter limitation is due to the character of the initial yield which depends on the magnitude of β. Numerical results for stresses and displacements are given in one case ($\beta = \frac{\pi}{4}$) for various positions of the elastic-plastic boundary.

INTRODUCTION

Although the general theory of plane strain for rigid, perfectly plastic material (unrestricted plastic flow) and its application have been successfully explored in recent years[1], complete solutions to problems of plane elastic-plastic strain (contained plastic deformation) have been possible in only a few cases, where the elastic-plastic boundary is determined a priori by symmetry considerations. References to existing solutions of plane elastic-plastic strain (in the theory of perfectly plastic solids) may be found in Prager and Hodge (1, Ch. 4 and 7), Hill (2, Ch. 5), and in a recent paper by Shaffer and House (3)[2]. Even for axi-symmetric problems, such as the thick-walled circular cylinder subjected to internal pressure (1, Ch. 4), a complete solution in closed form can be deduced only if the material is assumed to be incompressible in both the elastic and the plastic ranges; otherwise, the integration of the differential equations involved is accomplished by numerical methods.

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[1] See, for example, reference (1), Chapters 5 and 6; and reference (2), Chapters 6, 7, 8 and 9. Numbers in parentheses refer to the bibliography at the end of the paper.

[2] Available solutions obtained by total strain (deformation) theories of plasticity, as in (4), are ruled out here. Reference (4) was called to the author's attention by Professor W. Prager.

The present paper employing the stress-strain relations of Prandtl-Reuss, is concerned with the elastic, perfectly plastic solution of an incompressible isotropic wedge in the state of plane strain, subjected to a uniform normal pressure (assumed to be monotonically increasing with time) on one boundary only (Fig. 1). As the character of the initial yield and the locus of the elastic-plastic boundary depend on the nature of the load, as well as on the included wedge angle β, the present analysis is confined primarily to wedge regions where $\beta<\frac{\pi}{2}$. A complete solution (stresses and displacements) is given for a wedge region $0\leq\theta\leq\beta<\frac{\pi}{2}$ subjected to a uniform normal pressure, as shown in Fig. 1, and numerical results are plotted for the case when $\beta=\frac{\pi}{4}$. In the limit, the stress distribution reduces to the solution of the corresponding problem of a rigid, perfectly plastic material (1, pp. 156-158).

We recall that in the state of plane strain, here referred to cylindrical coordinates (r, θ, z), the radial and tangential displacements u_r and u_θ are at all times independent of z (the z-axis is taken perpendicular to the plane of the cross-section in Fig. 1), and that $u_z = 0$. The non-vanishing components of strain are given by

$$\varepsilon_r = \frac{\partial u_r}{\partial r}, \quad \varepsilon_\theta = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}, \quad 2\varepsilon_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \qquad [1]$$

and since $\varepsilon_z = 0$, it follows that

$$\varepsilon_r = -\varepsilon_\theta \qquad [2]$$

for an incompressible material. In the absence of body forces, the stress differential equations of equilibrium which are not identically satisfied may be written as

$$\frac{\partial S_r}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{S_r - S_\theta}{r} + \frac{\partial S}{\partial r} = 0 \qquad [3a]$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial S_\theta}{\partial \theta} + 2\frac{\sigma_{r\theta}}{r} + \frac{1}{r}\frac{\partial S}{\partial \theta} = 0 \qquad [3b]$$

where the components of the stress deviation tensor are related to the non-vanishing components of the stress tensor $\{\sigma_r, \sigma_\theta, \sigma_z, \sigma_{r\theta}\}$ by

$$S_r = \sigma_r - S \, , \; S_\theta = \sigma_\theta - S \, , \; S_z = \sigma_z - S = -(S_r + S_\theta) \qquad [4]$$

and the mean normal stress S , on account of the condition of incompressibility, is determined from the differential equations of equilibrium.

While equations [1] to [4] are valid in the elastic, as well as the plastic domain, the stress-strain relations in the elastic and plastic ranges are furnished respectively by Hooke's law, and the stress-strain law of Prandtl-Reuss which, for an incompressible material, reads

$$2G\left\{\dot{\varepsilon}_r, \dot{\varepsilon}_\theta, \dot{\varepsilon}_z, \dot{\varepsilon}_{r\theta}\right\} = \left\{\dot{S}_r, \dot{S}_\theta, \dot{S}_z, \dot{\sigma}_{r\theta}\right\} + \lambda\left\{S_r, S_\theta, S_z, \sigma_{r\theta}\right\} \qquad [5]$$

In [5], λ is a scalar factor which, in general, depends on the coordinates as well as time t , dot denotes differentiation with respect to time, and G is the shear modulus of elasticity. In addition, the Mises yield condition for the state of plane strain reduces to

$$S_r^2 + S_r S_\theta + S_\theta^2 + \sigma_{r\theta}^2 = k^2 \qquad [6]$$

where k is the yield limit in simple shear.

THE FULLY ELASTIC SOLUTION

The displacement field for an elastic incompressible isotropic material in the state of plane strain, referred to cylindrical coordinates (r, θ, z) is characterized by[3]

$$G\left[\nabla^2 u_r - \frac{1}{r}\left(\frac{u_r}{r} + \frac{2}{r}\frac{\partial u_\theta}{\partial \theta}\right)\right] + \frac{\partial S}{\partial r} = 0 \qquad [7a]$$

[3] For a discussion of the general equations of linear elasticity for an incompressible material, see (5, Section 3).

$$G\left[\nabla^2 u_\theta + \frac{1}{r}\left(\frac{2}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right)\right] + \frac{1}{r}\frac{\partial S}{\partial \theta} = 0 \qquad [7b]$$

$$\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) u_r + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0 \qquad [7c]$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$. The solution of [7] for a wedge subjected to uniform tractions may be written as

$$G u_r = -r(a \cos 2\theta + c \sin 2\theta) \qquad [8a]$$

$$G u_\theta = r(a \sin 2\theta - c \cos 2\theta) - d\, r \log r \qquad [8b]$$

$$S = 2(b + d\theta) \qquad [9]$$

from which, by [4] and Hooke's law, we have[4]

$$\begin{aligned} S_r &= -S_\theta = -2(a \cos 2\theta + c \sin 2\theta) \\ \sigma_{r\theta} &= -d + 2(a \sin 2\theta - c \cos 2\theta) \\ S_z &= 0 \end{aligned} \qquad [10]$$

For future purposes, we first record the values of the constants a , b , c , and d when solutions [9] and [10] are subjected to the boundary conditions

$$\begin{aligned} \sigma_\theta(\theta_1) &= \sigma_1 , \quad \sigma_{r\theta}(\theta_1) = \tau_1 \\ \sigma_\theta(\theta_2) &= \sigma_2 , \quad \sigma_{r\theta}(\theta_2) = \tau_2 \end{aligned} \qquad [11]$$

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[4] The components of the stress tensor, as given by [9] and [10], together with the special results [14], are given in (6, pp. 123-125).

and then deduce, as a special case, the values of the constants appropriate for a wedge of included angle $0 < \beta < 2\pi$, and uniformly loaded on one boundary. Thus, the constants of integration determined by [11] are

$$a = \frac{1}{D}\Big\{(\sigma_2-\sigma_1)\left[\cos 2\theta_2 - \cos 2\theta_1\right] + \tau_1\left[\sin 2\theta_1 - \sin 2\theta_2 + 2(\theta_2-\theta_1)\cos 2\theta_2\right] + \tau_2\left[\sin 2\theta_2 - \sin 2\theta_1 - 2(\theta_2-\theta_1)\cos 2\theta_1\right]\Big\}$$

$$c = \frac{1}{D}\Big\{(\sigma_2-\sigma_1)\left[\sin 2\theta_2 - \sin 2\theta_1\right] + \tau_1\left[\cos 2\theta_2 - \cos 2\theta_1 + 2(\theta_2-\theta_1)\sin 2\theta_2\right] + \tau_2\left[\cos 2\theta_1 - \cos 2\theta_2 - 2(\theta_2-\theta_1)\sin 2\theta_1\right]\Big\}$$

$$b = -\frac{1}{D}\Big\{\sigma_1\left[-1 + \cos 2(\theta_2-\theta_1) + 2\theta_2 \sin 2(\theta_2-\theta_1)\right] + \sigma_2\left[-1 + \cos 2(\theta_2-\theta_1) - 2\theta_1 \sin 2(\theta_2-\theta_1)\right] + \tau_1\left[-2\theta_1 - \sin 2(\theta_2-\theta_1) + 2\theta_2\cos 2(\theta_2-\theta_1)\right] + \tau_2\left[-2\theta_2 + \sin 2(\theta_2-\theta_1) + 2\theta_1\cos 2(\theta_2-\theta_1)\right]\Big\}$$

[12]

$$d = \frac{1}{D}\Big\{-2(\sigma_2-\sigma_1)\sin 2(\theta_2-\theta_1) + 2\tau_1\left[-1 + \cos 2(\theta_2-\theta_1)\right] + 2\tau_2\left[-1 + \cos 2(\theta_2-\theta_1)\right]\Big\}$$

where

$$D = 4\sin 2(\theta_2-\theta_1)\left[\tan(\theta_2-\theta_1) - (\theta_2-\theta_1)\right] \qquad [13]$$

For the special case shown in Fig. 1, i.e., when $\theta_1 = 0$, $\theta_2 = \beta$, $\sigma_1 = -p$, and $\sigma_2 = \tau_1 = \tau_2 = 0$, the above results reduce to

$$a = -p\,\frac{\tan\beta}{4\gamma}\,, \quad c = \frac{p}{4\gamma} \qquad [14]$$

$$b = -p\left(\frac{1}{2} + \frac{\tan\beta}{4\gamma}\right)\,, \quad d = -\frac{p}{2\gamma}$$

$$\gamma = \tan\beta - \beta$$

THE INITIAL YIELD AND THE ELASTIC-PLASTIC BOUNDARY

Since $S_r = -S_\theta$, the yield condition [6] becomes

$$S_\theta^2 + \sigma_{r\theta}^2 = k^2 \qquad [15]$$

which, by [10] and [14] for the wedge shown in Fig. 1, may be put in the form

$$f(\theta) = \left[\tan^2\beta + 2(1-\cos 2\theta) - 2\tan\beta\sin 2\theta\right] = \frac{k^2}{\left(p/2\gamma\right)^2} \qquad [16]$$

An examination of [16], together with its first and second derivatives, reveals that, depending on the magnitude of the wedge angle β, f assumes its maximum at different values of θ. Thus, in the range $0<\beta<2\pi$, plastic flow will first set in at one or more values of θ $(0\leq\theta\leq\beta)$, as given in Table 1. For example, in the range $0<\beta<\frac{\pi}{2}$, f is a mimimum at $\theta=\frac{\beta}{2}$, and will attain its maximum simultaneously at $\theta = 0$ and $\theta = \beta$. For $\beta=\frac{\pi}{2}$, the left-hand side of [16] is independent of θ and the entire region becomes plastic when the load reaches the value $(2\,k)$.

In the remainder of this paper, attention will be confined to wedge regions of which the included angle has the range $0 < \beta < \frac{\pi}{2}$. As seen in Table 1, yielding for this range of β will begin simultaneously at the outside surfaces when the pressure has reached the value

$$p^* = 2k\gamma(\tan\beta)^{-1} \qquad [17]$$

TABLE I

Values of θ for which f in [16], in the range $0 < \beta < 2\pi$, is a maximum.

Range of the Wedge Angle	Values of θ, where Yielding Will Initiate
$0 < \beta < \frac{\pi}{2}$	$\theta = 0, \beta$
$\beta = \frac{\pi}{2}$	all $0 \le \theta \le \frac{\pi}{2}$
$\frac{\pi}{2} < \beta < \frac{3\pi}{2}$	$\theta = \frac{\beta}{2}$
$\beta = \frac{3\pi}{2}$	all $0 \le \theta \le \frac{3\pi}{2}$
$\frac{3\pi}{2} < \beta < 2\pi$	$\theta = \frac{\beta}{2} \pm \frac{\pi}{2}$

For a pressure $p > p^*$, a portion of the wedge region becomes plastic, and since the yield condition is independent of r , the elastic-plastic boundary must be a wedge bounded by $\theta = \phi_1$ and $\theta = \phi_2$, as indicated in Fig. 1.

THE ELASTIC-PLASTIC SOLUTION

In the elastic region $\phi_1 \leq \theta \leq \phi_2$, the solution is still of the form [8], [9], and [10], but the boundary conditions are now

$$S_\theta^2(\phi_1) + \sigma_{r\theta}^2(\phi_1) = k^2$$
$$S_\theta^2(\phi_2) + \sigma_{r\theta}^2(\phi_2) = k^2 \qquad [18]$$

as well as the conditions for conditions for continuity of stresses in the form [11] with θ_1 and θ_2 replaced respectively by ϕ_1 and ϕ_2. With the use of the latter four conditions, the coefficients a , b , c , and d (as functions of ϕ_1 and ϕ_2) may still be written in the form /12/, where now σ_1 , τ_1 , σ_2, and τ_2 refer to the normal and shearing components of stress at the elastic-plastic boundaries. Here, we also note that according to [10], [8], and [2], at a generic point P in the elastic domain

$$S_r : S_\theta : S_z = \varepsilon_r : \varepsilon_\theta : \varepsilon_z = 1 : -1 : 0 \quad ; \quad \phi_1 \leq \theta \leq \phi_2 \qquad [19]$$

Since by [9] and [10], the components of stress in the elastic region do not involve r , continuity of the stresses at the elastic-plastic boundaries demands that these quantities remain independent of r throughout the wedge. Hence, the stress differential equations of equilibrium [3] reduce to

$$\frac{\partial \sigma_{r\theta}}{\partial \theta} + (S_r - S_\theta) = 0 \qquad /20a/$$

$$\frac{\partial S_\theta}{\partial \theta} + 2\sigma_{r\theta} + \frac{\partial S}{\partial \theta} = 0 \qquad /20b/$$

which hold in the elastic, as well as the plastic regions $0 \leq \theta \leq \phi_1$ and $\phi_2 \leq \theta \leq \beta$. Differentiating the first of the above with respect to t , combining (after multiplication by λ) the resulting equation with /20a/, and

making use of the stress-strain relations of Prandtl-Reuss, and the incompressibility relation [7c], there results the following differential equation for the radial displacement in the plastic domain:

$$\frac{1}{r}\frac{\partial^2 \dot{u}_r}{\partial \theta^2} - r\frac{\partial^2 \dot{u}_r}{\partial r^2} + 3\frac{\partial \dot{u}_r}{\partial r} + \frac{\dot{u}_r}{r} = \frac{\sigma_{r\theta}}{G}\frac{\partial \lambda}{\partial \theta} \quad . \qquad [21]$$

Since the loading is assumed to increase monotonically with time, as in previous elastic-plastic investigations, it is convenient to interpret the dot as indicating differentiation with respect to ϕ, which, when necessary, will be identified with either ϕ_1 or ϕ_2.

As radial slip is not permissible along the elastic-plastic boundaries, it follows that the displacement u_r must be a linear function of r also in the plastic domain, or simply of the form

$$u_r = r\,\varepsilon_r(\theta, \phi) \qquad [22]$$

Substitution of [22] into [21], followed by an integration with respect to ϕ, yields

$$\frac{\partial^2 \varepsilon_r}{\partial \theta^2} + 4\varepsilon_r = \int_0^{\phi} \frac{\sigma_{r\theta}}{G}\frac{\partial \lambda}{\partial \theta}\, dt \quad , \qquad [23]$$

the homogeneous solution of which, except for a factor of r, is of the form [8a]. Continuity of ε_r (or u_r) at the elastic-plastic boundaries requires the vanishing of any particular solution of [23] which, since $\sigma_{r\theta} \neq 0$, is equivalent to $\frac{\partial \lambda}{\partial \theta} = 0$, or

$$\lambda = \lambda(\phi) \qquad [24]$$

Thus, by [22] and [7c], the displacements u_r and u_θ in both the elastic and plastic domains are of the form [8a, b], which, together with

[19], imply the truth of

$$\varepsilon_r : \varepsilon_\theta : \varepsilon_z = \dot{\varepsilon}_r : \dot{\varepsilon}_\theta : \dot{\varepsilon}_z = 1 : -1 : 0 \quad ; \qquad 0 < \phi_1 < \phi_2 < \beta \, , \qquad [25]$$

at P, for all states of contained plastic deformation. Moreover, in view of [19], [24], [25], and the continuity of stresses at ϕ_1 and ϕ_2, the stress-strain law of Prandtl-Reuss [6] demands that at

$$S_r : S_\theta : S_z = \dot{S}_r : \dot{S}_\theta : \dot{S}_z = 1 : -1 : 0 \quad ; \qquad 0 < \phi_1 < \phi_2 < \beta \, . \qquad [26]$$

According to [25], S_r and S_θ have the same absolute value throughout the wedge. Hence, the quantity $(S_r - S_\theta) = -2S_\theta$ and, with the aid of the Mises yield condition, [20a] in the plastic domain becomes

$$\frac{\partial \sigma_{r\theta}}{\partial \theta} = \pm 2 \left[k^2 - \sigma_{r\theta}^2 \right]^{1/2} \qquad [27]$$

Integration of [27] together with [15] and [26] result in

$$\begin{aligned} \sigma_{r\theta}(\theta, \phi) &= \pm k \sin 2(\theta - \gamma) \\ S_\theta(\theta, \phi) = -S_r(\theta, \phi) &= \pm k \cos 2(\theta - \gamma) \end{aligned} \qquad [28]$$

where $\gamma = \gamma(\phi)$, and the question of the appropriate signs in each of the two plastic regions will be disposed of presently. The mean normal stress, in the plastic domain, is determined by substituting [28] into [20b], which leads to $\frac{\partial S}{\partial \theta} = 0$, or

$$S = S(\phi) \; . \qquad [29]$$

The stress tensor in the plastic regions $0 \leq \theta \leq \phi_1$ and $\phi_2 \leq \theta \leq \beta$, in addition to fulfilling the continuity requirements at ϕ_1 and ϕ_2, must also conform to the boundary conditions

$$\sigma_\theta(0, \phi_1) = -p \quad , \quad \sigma_{r\theta}(0, \phi_1) = 0$$
$$\sigma_\theta(\beta, \phi_2) = \sigma_{r\theta}(\beta, \phi_2) = 0 \qquad [30]$$

respectively. Substitution of [28] and [29] into each of the conditions [30] yields, respectively

$$\gamma = 0 \quad , \quad S = -p \mp k \quad ; \quad 0 \leq \theta \leq \phi_1 \qquad [31a]$$

and

$$\gamma = \beta \quad , \quad S = \mp k \quad ; \quad \phi_2 \leq \theta \leq \beta \quad , \qquad [31b]$$

in each of the two plastic regions. Guided by the fact[5] that the lateral pressure bends the wedge so as to produce a tensile radial stress on $\theta = 0$, and a compressive radial stress on $\theta = \beta$, we choose the lower sign in [28] and [31a] for the plastic region $0 \leq \theta \leq \phi_1$, and the upper sign in [28] and [31b] for the region $\phi_2 \leq \theta \leq \beta$. In this manner, we finally obtain the following stress distribution in the two plastic regions:

$$\left.\begin{aligned} S_r &= -S_\theta = k \cos 2\theta \;, \quad S_z = 0 \\ \sigma_{r\theta} &= -k \sin 2\theta \\ S &= -p + k \end{aligned}\right\} ; \quad 0 \leq \theta \leq \phi_1 \qquad [32]$$

and

$$\left.\begin{aligned} S_r &= -S_\theta = -k \cos 2(\theta - \beta) \,, \quad S_z = 0 \\ \sigma_{r\theta} &= k \sin 2(\theta - \beta) \\ S &= -k \end{aligned}\right\} ; \quad \phi_2 \leq \theta \leq \beta \qquad [33]$$

[5] This remark is similar to that offered in (1, p. 157) for a wedge of rigid, perfectly plastic material.

Continuity of the stresses at the elastic-plastic boundaries, together with boundary conditions [18] for the stresses in the elastic region $\phi_1 \leq \theta \leq \phi_2$, require the continuity of the mean normal stress at ϕ_1 and ϕ_2. Hence, by [10], [32] and [33], we have

$$
\begin{aligned}
2(b + d\phi_1) &= -p + k \\
2(b + d\phi_2) &= -k
\end{aligned}
\qquad [34]
$$

where b and d are of the form given in [12], with θ_1 and θ_2 replaced by ϕ_1 and ϕ_2 ; also, by [32] and [33], the quantities σ_1, τ_1, σ_2, and τ_2 in [12] are

$$\sigma_1 = -p + k(1 - \cos 2\phi_1) \quad , \quad \tau_1 = -k \sin 2\phi_1 \qquad [35a]$$

and

$$\sigma_2 = -k\left[1 - \cos 2(\phi_2 - \beta)\right] \quad , \quad \tau_2 = k \sin 2(\phi_2 - \beta) \qquad [35b]$$

Elimination of p from [34] leads to a single equation which, after considerable manipulation, reduces to

$$\phi_1 + \phi_2 = \beta \qquad [36]$$

revealing the interdependence of ϕ_1 and ϕ_2. Thus, in what follows, one of the elastic-plastic boundaries, i.e., ϕ_1, will be identified with ϕ, and the other, ϕ_2, with $(\beta - \phi)$.

In summary, a statement of the complete solution of the problem is as follows: The displacements u_r and u_θ throughout the wedge ($0 \leq \theta \leq \beta$) are of the form [8a, b]; the stresses in the elastic region $\phi \leq \theta \leq (\beta - \phi)$ are of the form [10] and [9]; the stresses in the two plastic regions $0 \leq \theta \leq \phi$ and $(\beta - \phi) \leq \theta \leq \beta$ are given, respectively, by [32] and [33]; and the coefficient functions a , b , c , and d are of the form [12], with σ_1, τ_1, σ_2, τ_2 given by [35], and θ_1 and θ_2 (as well as ϕ_1 and ϕ_2) replaced, respectively, by ϕ and $(\beta - \phi)$. Also, the functional relation between the

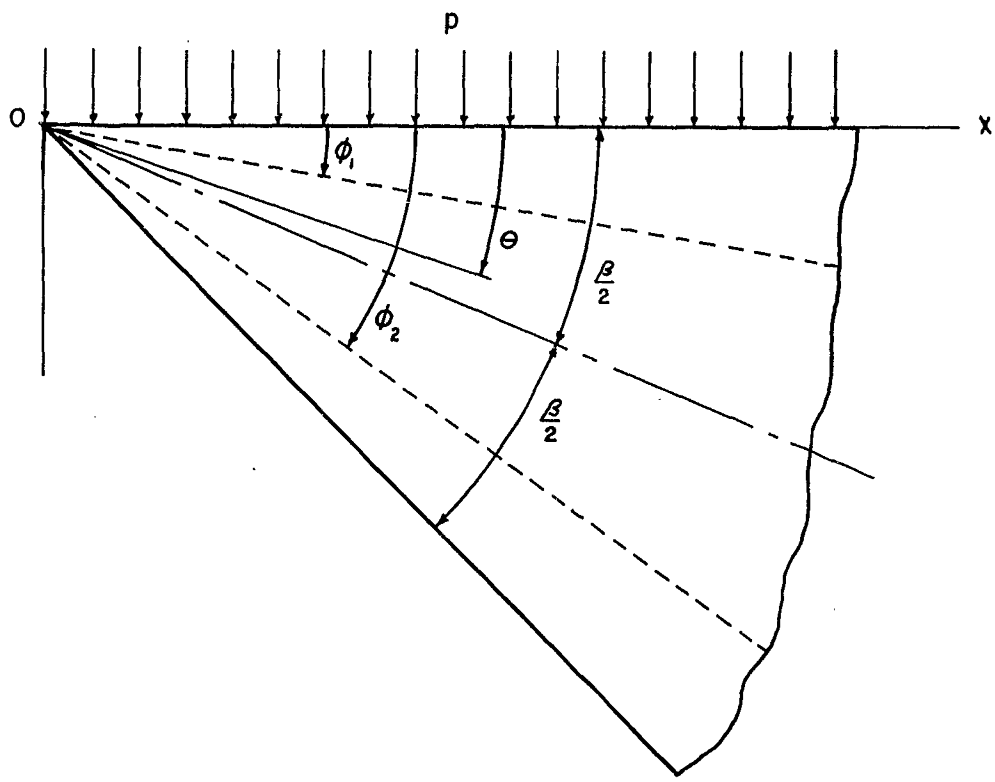

Fig. 1: Narrow wedge under a uniform lateral pressure.

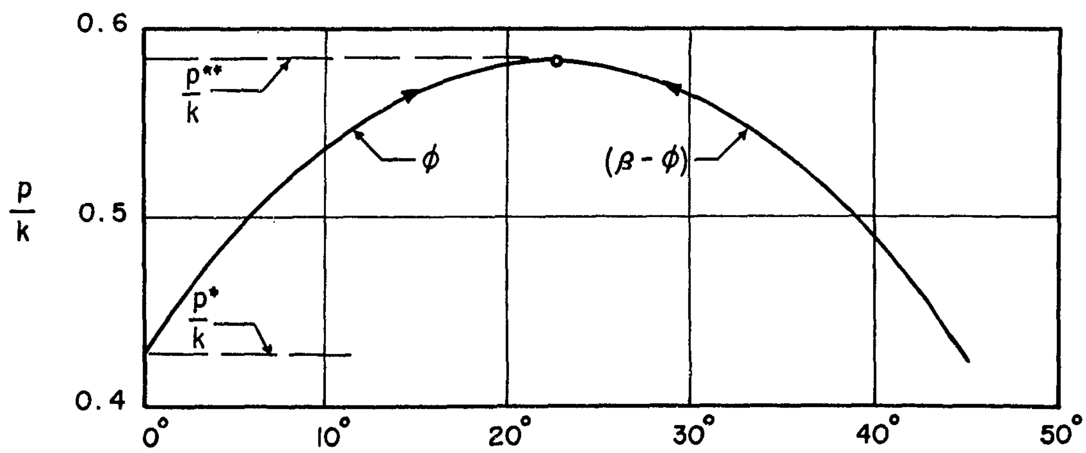

Fig. 2: Lateral pressure p versus elastic-plastic boundaries, $0 < \phi < \frac{\beta}{2}$ and $\frac{\beta}{2} < (\beta - \phi) < \beta$, for $\beta = \frac{\pi}{4}$

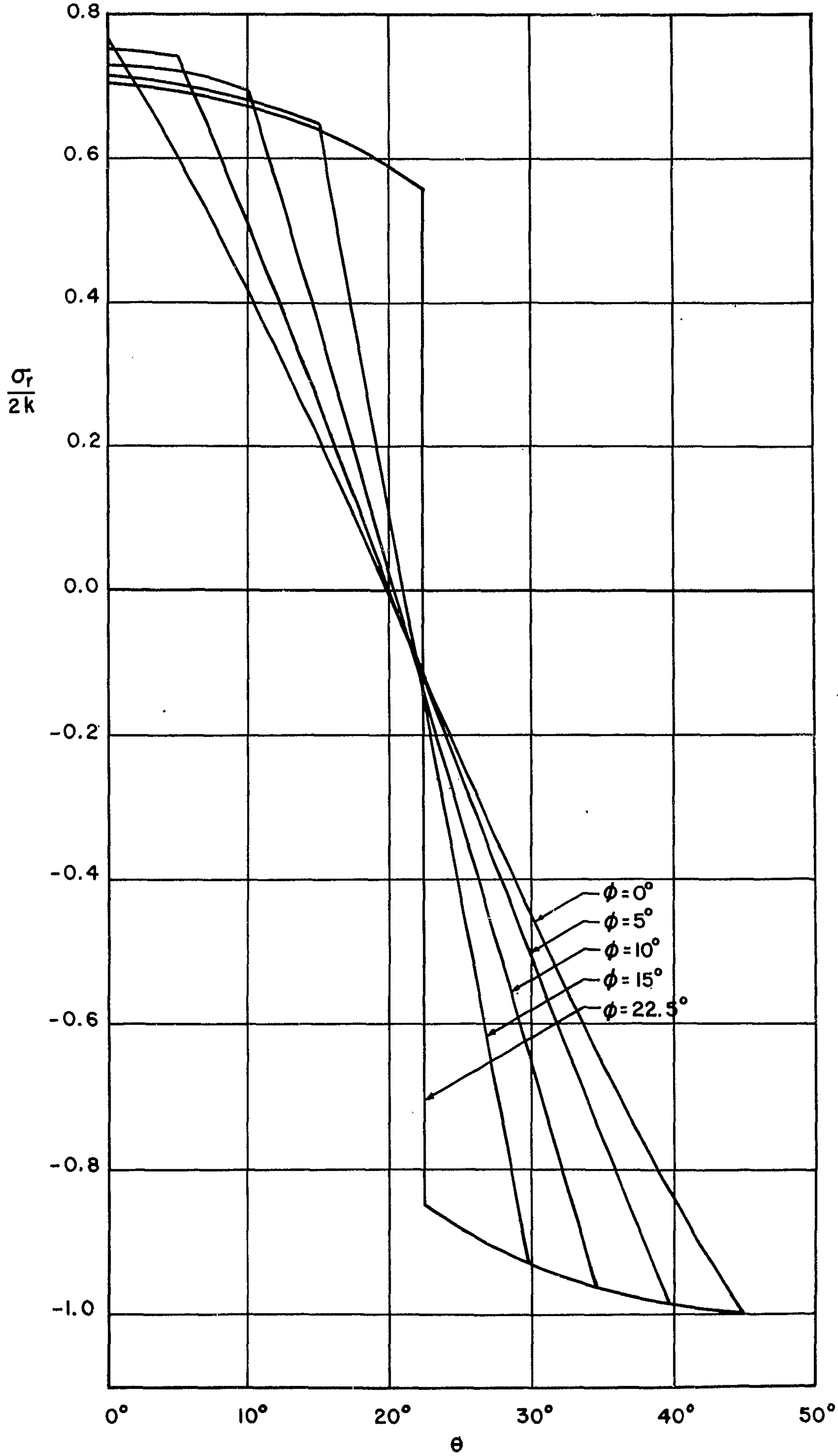

Fig. 3: Distribution of radial stress.

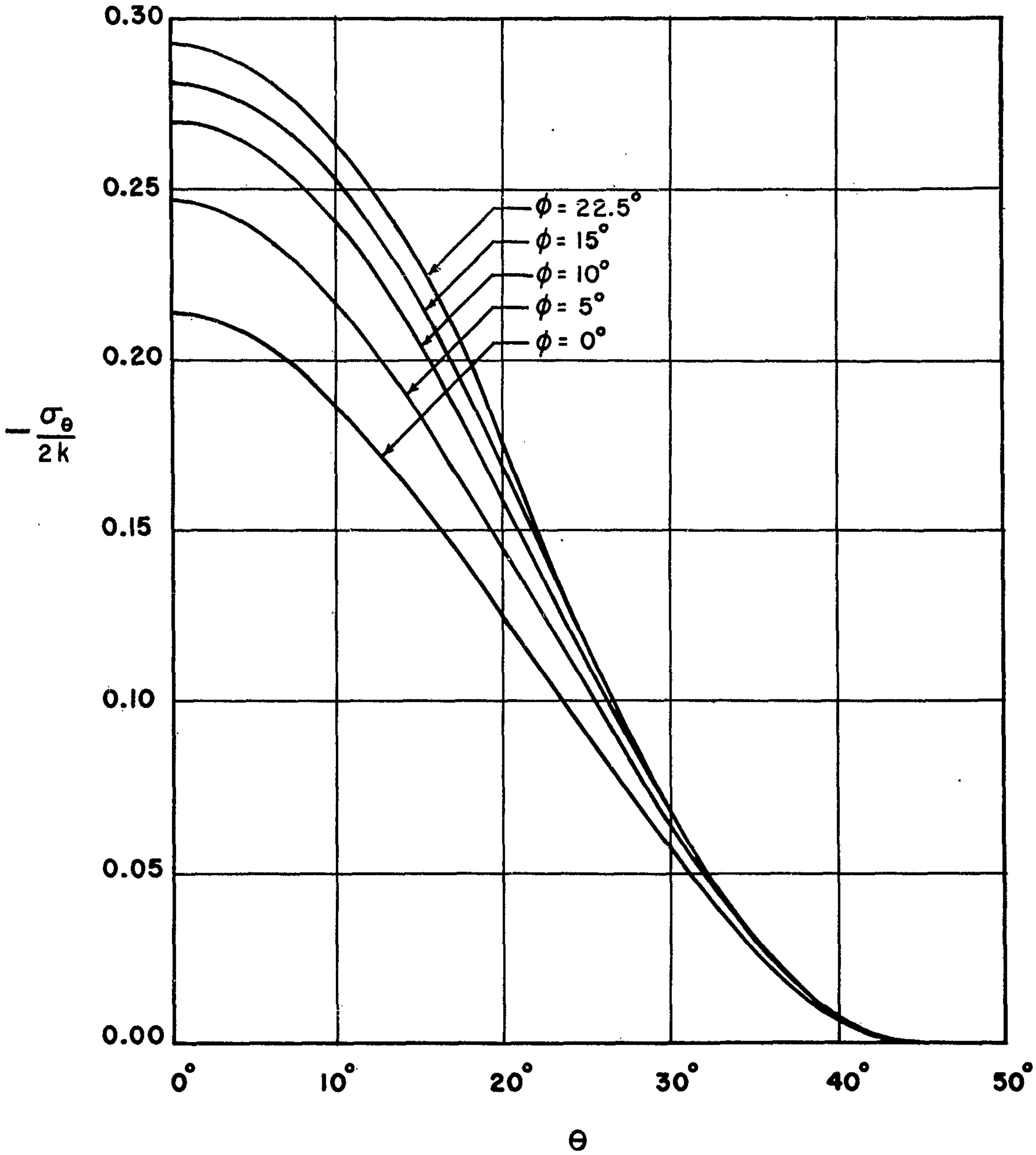

Fig. 4: Distribution of circumferential stress.

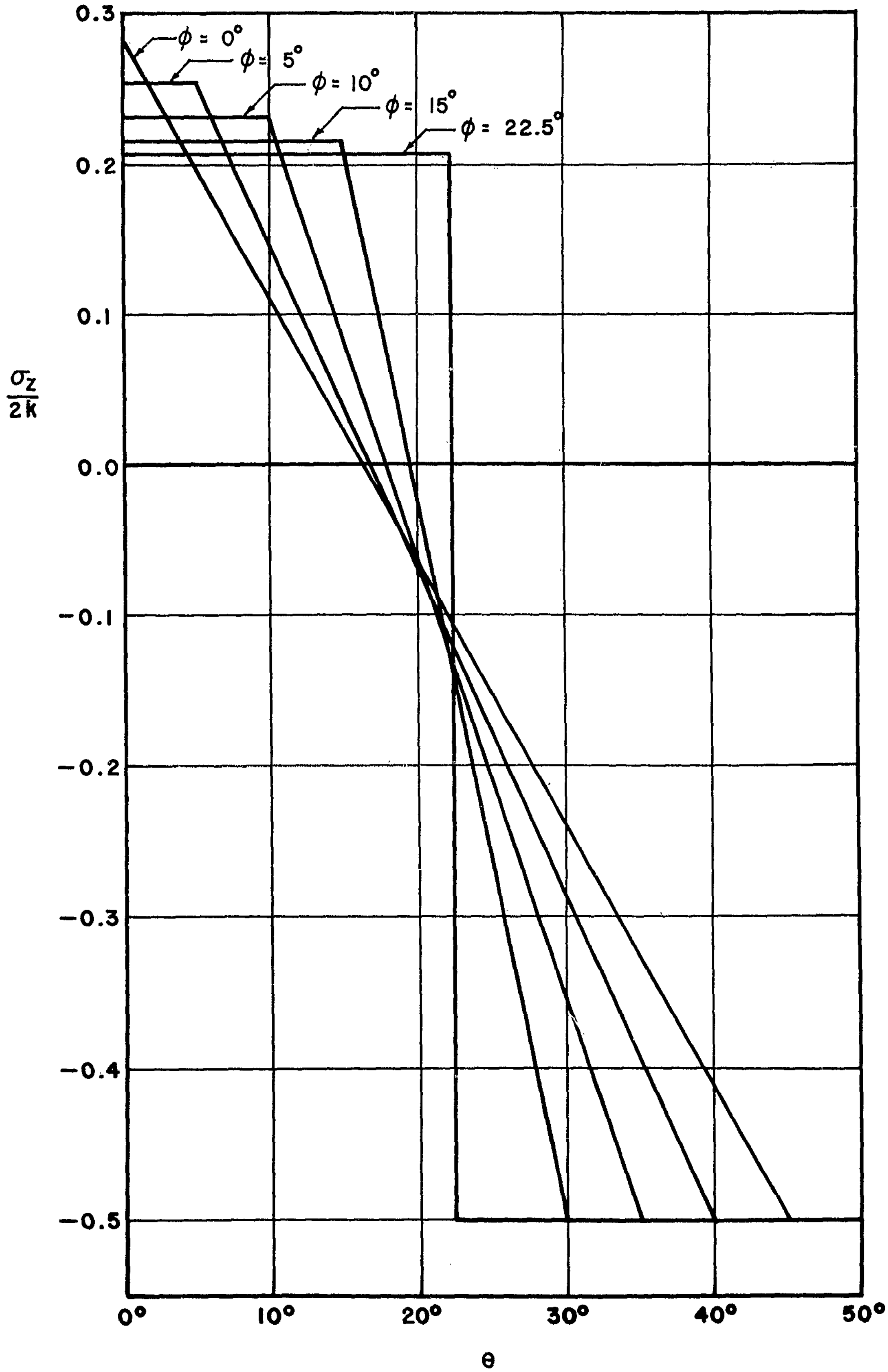

Fig. 5: Distribution of axial stress.

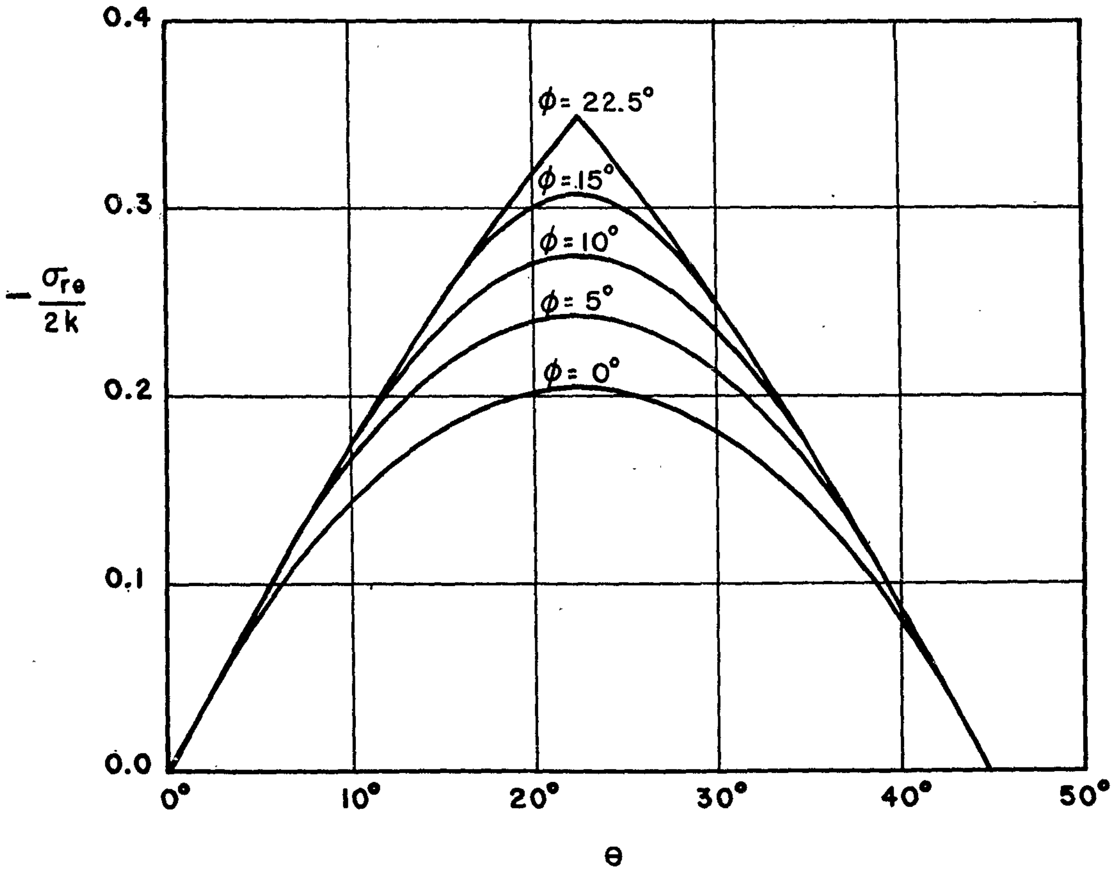

Fig. 6: Distribution of shearing stress.

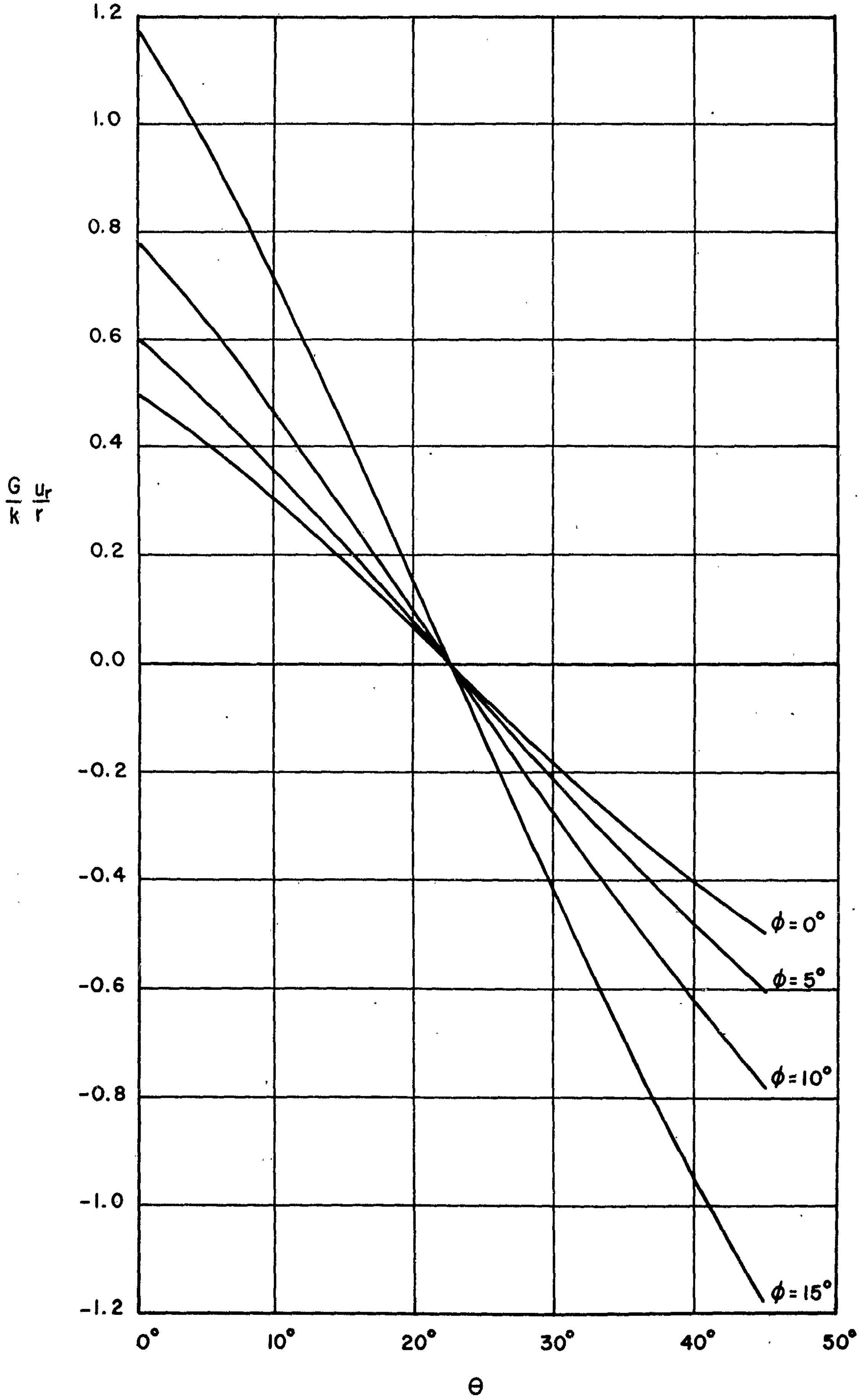

Fig. 7: Ratio of radial displacement to r (or radial strain)

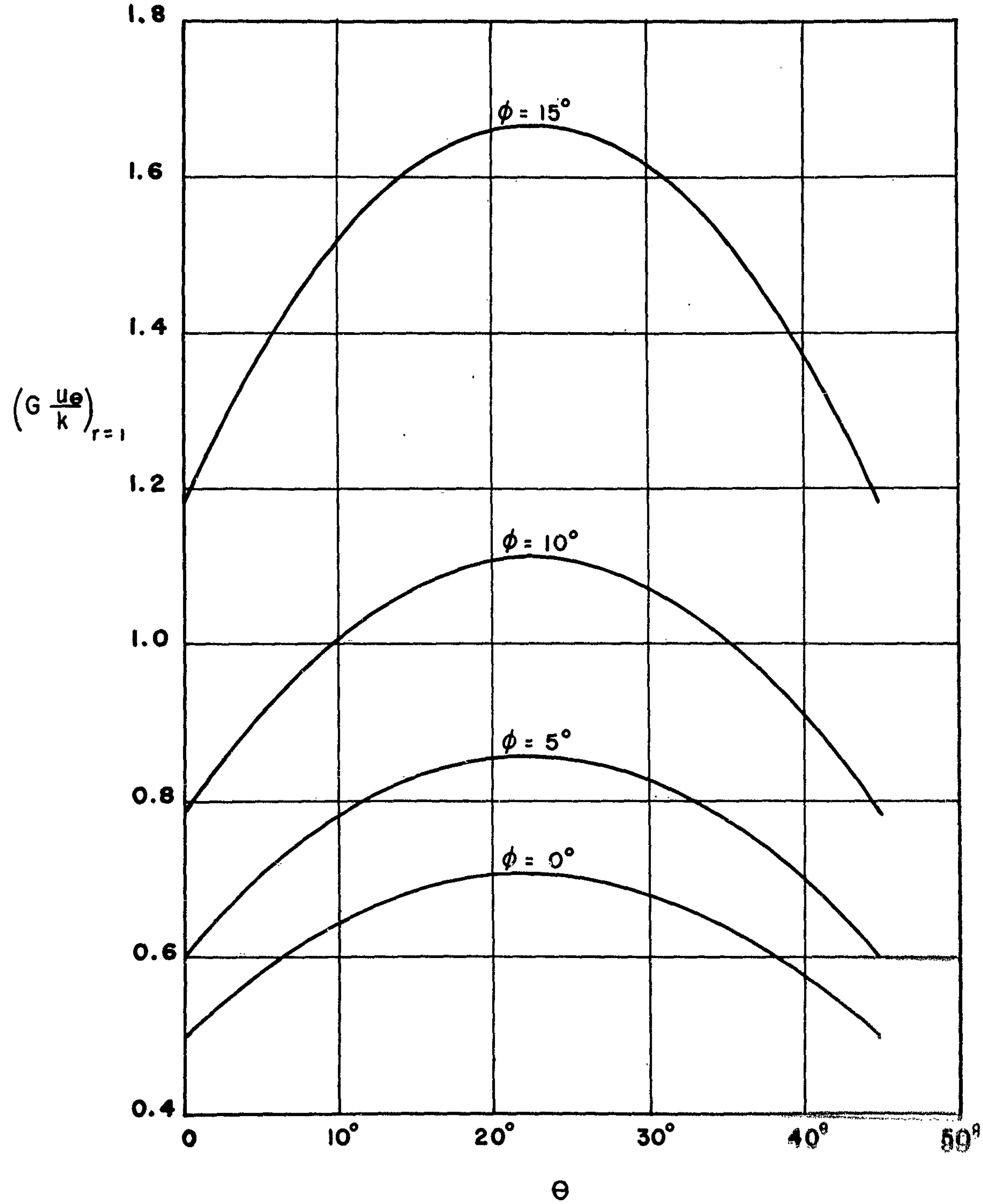

Fig. 8: Circumferential displacement at r = 1.

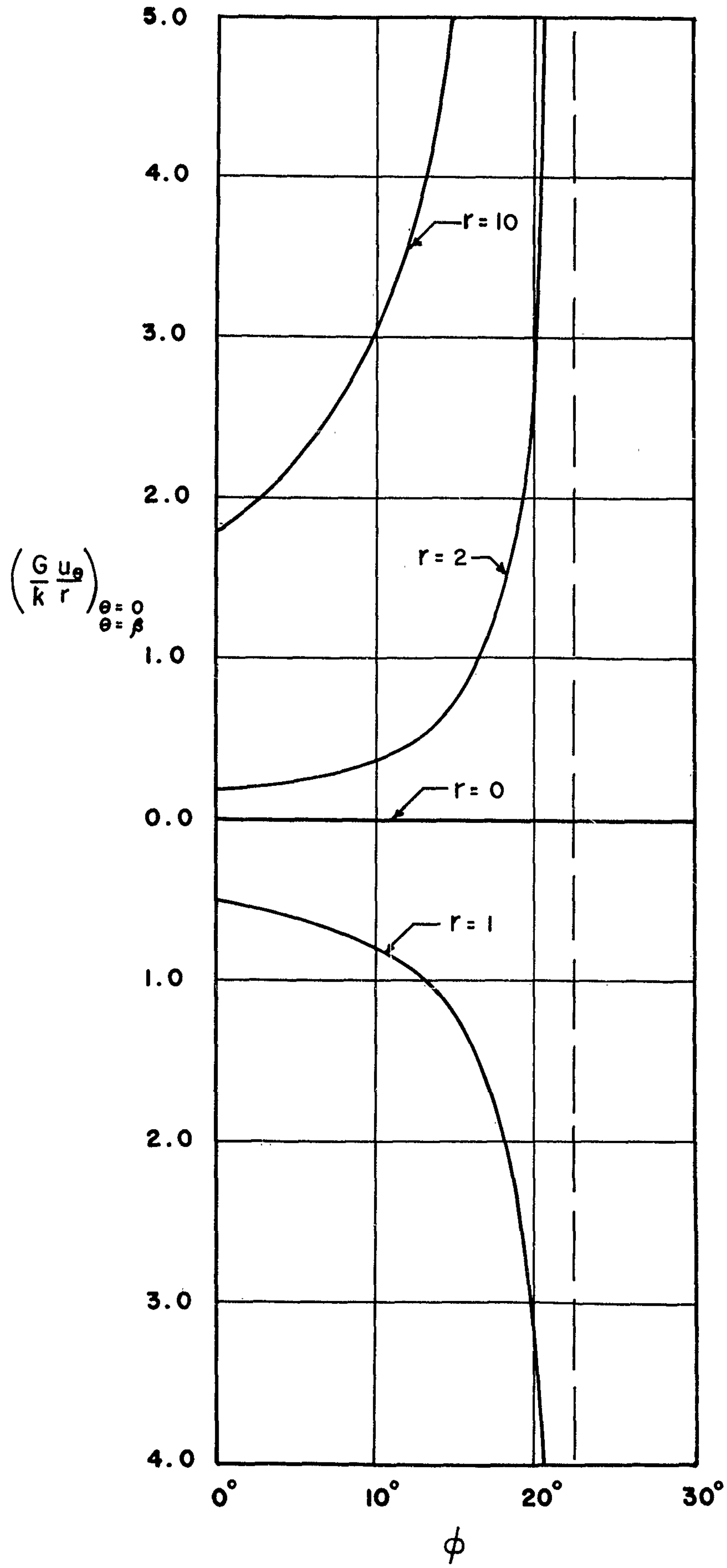

Fig. 9: Radio of circumferential displacement to r at various positions along the outside boundaries.

lateral pressure p and the elastic-plastic boundary $\theta = \phi$, deduced with the aid of [36] from either of [34], is given by

$$\frac{p}{k} = \frac{2}{\tan\alpha}\left[\alpha \sin 2\phi(\tan\alpha + \tan\phi) + (\tan\alpha - \alpha)\right] \qquad [37]$$

$$\alpha = \beta - 2\phi$$

In the limit of fully plastic state, the components of stress in the two plastic regions, given by [32] and [33], reduce to the known results for a wedge of a rigid, perfectly plastic material, where $\phi = \phi_1 = \phi_2 = \frac{\beta}{2}$ (separating the two plastic regions) is a line of stress discontinuity.

NUMERICAL EXAMPLE - CONCLUSION

We now consider a numerical example and assume $\beta = \frac{\pi}{4}$. For this particular case, Fig. 2 shows the values of the elastic-plastic boundaries ϕ and $(\beta - \phi)$ corresponding to given values of $\frac{p}{k}$ which, in the range of contained plastic deformation, is bounded by $\frac{p^*}{k}$ and $\frac{p^{**}}{k}$. The resulting stress distributions and displacements, in this case ($\beta = \frac{\pi}{4}$), for various positions of ϕ are shown in Figs. 3 to 9.

In conclusion, it may be mentioned that while the validity of the foregoing elastic-plastic solution is limited to wedge regions with included angle $\beta < \frac{\pi}{2}$, many features of the analysis presented are also applicable to other ranges of values of β and may be extended to cover wedge regions exposed to more general uniform surface tractions.

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